

ON A PROBLEM BY R. SALEM CONCERNING MINKOWSKI'S QUESTION MARK FUNCTION

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ABSTRACT. Minkowski's question mark function is strictly increasing on $[0, 1]$ and hence defines a Stieltjes measure on $[0, 1]$. A problem originating from Salem in 1943, is to determine whether the Fourier series of this measure decay to zero or not. The purpose of this note is to mention that a recent result by Jordan and Sahlsten implies that the Fourier transform decays to zero with a polynomial speed.

Minkowski's question mark function $?$ was introduced by H. Minkowski in order to give an example of a one-to-one map between the rational in $(0, 1)$ and the quadratic irrationals in $(0, 1)$. R. Salem showed in [4] that

$$?(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2^{a_1(x)+a_2(x)+\dots+a_k(x)-1}},$$

where $[0; a_1(x), a_2(x), \dots]$ is the continued fraction expansion of $x \in [0, 1]$. The function $?: [0, 1] \rightarrow [0, 1]$ is strictly increasing and so it defines a Stieltjes measure μ of mass one. It was mentioned in [4] that it does not seem to be known whether the Fourier series of μ decays to zero at infinity.

According to J. R. Kinney [2] the Hausdorff dimension of the measure μ is given by

$$\dim_{\mathrm{H}} \mu = \left(2 \int_0^1 \log_2(1+x) d\mu(x) \right)^{-1}.$$

Since $0 < \log_2(1+x) < 1$ on $(0, 1)$ we clearly have $\dim_{\mathrm{H}} \mu > 1/2$.

The measure μ can also be obtained in the following equivalent way, as mentioned by F. Canto-Martín, H. Hedenmalm, and A. Montes-Rodríguez on page 43 of [3]. Let $G: [0, 1) \rightarrow [0, 1)$ be the Gauß-map, defined by

$$G: x \mapsto 1/x \pmod{1}.$$

The map G is continuous and decreasing on each of the intervals

$$I_k = (1/(k+1), 1/k], \quad k = 1, 2, 3, \dots$$

The measure μ is then the G -invariant Bernoulli measure obtained by defining

$$\mu(I_k) = 2^{-k}.$$

The problem concerning the Fourier series of μ mentioned by R. Salem is also mentioned in the above mentioned paper by F. Canto-Martín, H. Hedenmalm, and A. Montes-Rodríguez.

Since $\dim_{\mathrm{H}} \mu > 1/2$, as mentioned above, it is an immediate consequence of the result of T. Jordan and T. Sahlsten [1, Theorem 1.3] that there is a number $\eta > 0$ such that

$$|\hat{\mu}(\xi)| = O(|\xi|^{-\eta}) \quad \text{as } |\xi| \rightarrow \infty.$$

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Hence, the theorem by Jordan and Sahlsten solves the problem raised by R. Salem.

I noticed this on January 1, 2015 and told T. Jordan and T. Sahlsten about it the same day. The day before, December 31, 2014, another solution to Salem's problem was submitted to Arxiv by S. Yakubovich [5], and appeared on Arxiv on January 5, 2015. Before January 5, I did not know about Yakubovich's result. The paper by T. Jordan and T. Sahlsten, from which the result immediately follows, appeared more than one year before, on December 12, 2013.

REFERENCES

- [1] T. Jordan, T. Sahlsten, *Fourier transforms of Gibbs measures for the Gauss map*, preprint, arXiv:1312.3619
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- [4] R. Salem, *On some singular monotonic functions which are strictly increasing*, Transactions of the American Mathematical Society 53 (1943), 427–439.
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